Shortness coefficient of cyclically 4-edge-connected cubic graphs

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Submitted: Jan 10, 2019; Accepted: Jan 20, 2020; Published: Feb 7, 2020

Abstract

Grünbaum and Malkevitch proved that the shortness coefficient of cyclically 4-edge-connected cubic planar graphs is at most $\frac{76}{77}$. Recently, this was improved to $\frac{359}{366}(< \frac{52}{53})$ and the question was raised whether this can be strengthened to $\frac{41}{42}$, a natural bound inferred from one of the Faulkner-Younger graphs. We prove that the shortness coefficient of cyclically 4-edge-connected cubic planar graphs is at most $\frac{37}{38}$ and that we also get the same value for cyclically 4-edge-connected cubic graphs of genus g for any prescribed genus $g \geqslant 0$. We also show that $\frac{45}{46}$ is an upper bound for the shortness coefficient of cyclically 4-edge-connected cubic graphs of genus g with face lengths bounded above by some constant larger than 22 for any prescribed $g \geqslant 0$.

Mathematics Subject Classifications: 05C38, 05C45, 05C10

*Supported by the grant SCHM 3186/2-1 (401348462) from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) and by DAAD, Germany (as part of BMBF) and by the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57320575.

†Supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO).

1 Introduction

In 1973, Grünbaum and Walther [12] introduced two limits called *shortness coefficient* and *shortness exponent* that measure how far a given infinite family $\mathcal{G}$ of graphs is from being Hamiltonian. Formally, the shortness coefficient of $\mathcal{G}$ is defined as

$$\rho(\mathcal{G}) := \liminf_{G\in\mathcal{G}} \frac{\operatorname{circ}(G)}{|V(G)|}$$

and the shortness exponent of $\mathcal{G}$ as

$$\sigma(\mathcal{G}) := \liminf_{G\in\mathcal{G}} \frac{\log \operatorname{circ}(G)}{\log |V(G)|},$$

where the *circumference* $\operatorname{circ}(G)$ denotes the length of a longest cycle in a given graph G. Clearly, for every infinite family $\mathcal{G}$ of graphs, $\sigma(\mathcal{G}) < 1$ implies $\rho(\mathcal{G}) = 0$.

Tutte's celebrated result that 4-connected planar graphs are Hamiltonian [26] implies therefore that the shortness coefficient of the 4-connected planar graphs is 1, and the same conclusion holds if we relax the prerequisite of 4-connectedness to 'containing at most three 3-vertex-cuts' [4]—for a more detailed overview of hamiltonicity in planar graphs with few 3-vertex-cuts we refer the reader to [22]. However, it is well-known that infinitely many non-Hamiltonian graphs appear when sufficiently many 3-vertex-cuts are present: Moon and Moser [20] showed that the shortness exponent of the 3-connected planar (and even maximal planar) graphs is at most $\log_3 2$, while Chen and Yu [5] showed that this upper bound is tight, i.e. the shortness exponent of these graphs is $\log_3 2$. This implies that the shortness coefficient of the 3-connected planar graphs (that is, the 1-skeleta of polyhedra [23]) is 0.

Historically, key results in the theory of Hamiltonicity have proven that connectivity and circumference of a graph are intimately linked. In the study of cubic graphs, the classic vertex- and edge-connectivity notions are only of limited use—instead, the following more fine-grained connectivity notion has been established: A graph G is *cyclically k-edge-connected*[1] if, for every edge-cut S of G with less than k edges, at most one component of $G - S$ contains a cycle. For a positive integer k, let $\mathcal{C}k$ be the class of connected cyclically k-edge-connected cubic graphs, and let $\mathcal{C}k\mathcal{P}$ be the subclass of planar graphs in $\mathcal{C}k$. It is well known that every graph in $\mathcal{C}k$ is $\min\{k, 3\}$-connected. Cyclically 4-edge-connected cubic graphs thus have connectivity 3 but inherit some properties of 4-connected graphs; in the light of the preceding paragraph, an important question is therefore whether the shortness coefficient of $\mathcal{C}4\mathcal{P}$ is strictly between 0 and 1.

Aldred, Bau, Holton, and McKay [1] showed that the smallest non-Hamiltonian members in $\mathcal{C}4\mathcal{P}$ have 42 vertices and that there are exactly three such graphs up to isomorphism, including the Grinberg graph [10] and one of the Faulkner-Younger graphs [8]. As Thomassen writes [24], Tutte's theorem [26] implies that any n-vertex graph $G \in \mathcal{C}4\mathcal{P}$

[1] For cubic graphs in the literature frequently shortened to its vertex-analogue *cyclically k-connected*, as in the cubic case the two terms coincide if we exclude the triangular prism.

has a cycle such that the vertices not in that cycle are pairwise non-adjacent. Since any such cycle must contain at least 3/4 of the vertices of G, $\mathrm{circ}(G) \geqslant \frac{3}{4}n$.[2] By constructing a graph H from parts of the 42-vertex Grinberg graph and replacing every vertex of a 4-regular 4-connected planar graph with a copy of H, Grünbaum and Malkevitch [11] showed that there are infinitely many n-vertex graphs in $\mathcal{C}4\mathcal{P}$ with circumference at most $\frac{76}{77}n$, which gives $\rho(\mathcal{C}4\mathcal{P}) \leqslant \frac{76}{77}$.[3] Recently, the first and second authors [17] improved this long standing upper bound to $\rho(\mathcal{C}4\mathcal{P}) < \frac{52}{53}$, and raised the question whether $\rho(\mathcal{C}4\mathcal{P}) \leqslant \frac{41}{42}$ holds, which is inspired by the fact that the smallest non-Hamiltonian graphs in $\mathcal{C}4\mathcal{P}$ have 42 vertices and circumference 41 each.

Here, we show that this is the case by proving $\rho(\mathcal{C}4\mathcal{P}) \leqslant \frac{37}{38}$. We achieve this bound by using a construction of graphs whose largest face length goes to infinity (where the *length* of a face is defined to be the length of the shortest closed walk bounding the face). As a natural follow-up question, one might ask whether such a construction is still possible when all face lengths are bounded by a constant. This is indeed the case, as we shall prove for the graphs in $\mathcal{C}4\mathcal{P}$ whose face lengths are at most some constant which is larger than 22 that the shortness coefficient is at most $\frac{45}{46}$.

Bondy and Simonovits [2] showed that $\sigma(\mathcal{C}3) \leqslant \log_9 8 \approx 0.946$, while Liu, Yu, and Zhang [16] showed that $\sigma(\mathcal{C}3) \geqslant 0.8$. Walther [27] proved that $\sigma(\mathcal{C}3\mathcal{P}) \leqslant \log_{27} 26$ (see also Theorem B in [12]), which solves an open problem by Grünbaum and Motzkin. Harant [14, 15] and Owens [21] proved for various subclasses of $\mathcal{C}3\mathcal{P}$ having at most two different face sizes that their respective shortness exponents are less than 1. Hence, the shortness coefficients $\rho(\mathcal{C}3)$ and $\rho(\mathcal{C}3\mathcal{P})$ of the 3-connected cubic graphs and the 3-connected cubic planar graphs are 0.

In stark contrast, the precise value of $\rho(\mathcal{C}4)$ is not known. Indeed, the famous conjecture of Thomassen that every 4-connected line-graph is Hamiltonian [25] is equivalent to the statement that every n-vertex graph in $\mathcal{C}4$ has a dominating cycle [9], and an affirmative answer to this would in turn imply a lower bound of $\frac{3}{4}n$ on the circumference of these graphs. More conservatively, Bondy (see [9]) has conjectured that there is a constant $0 < c < 1$ such that the circumference of every n-vertex graph in $\mathcal{C}4$ is at least cn. This would imply $\rho(\mathcal{C}4) \geqslant c > 0$, while Máčajová and Mazák [19] even conjecture $\rho(\mathcal{C}4) \geqslant c \geqslant \frac{7}{8}$, and Markström [18] conjectures that $\rho(\mathcal{C}4) = 0$.

Despite the lack of non-trivial lower bounds for $\rho(\mathcal{C}4)$, an upper bound for $\rho(\mathcal{C}4)$ is known: Máčajová and Mazák [19] showed recently that $\mathcal{C}4$ contains an infinite graph family in which the circumference of every n-vertex graph is at most $\frac{7}{8}n$, which implies $\rho(\mathcal{C}4) \leqslant \frac{7}{8}$. Here, we provide a general theorem (Theorem 7) that implies the result of [19]. We extend our results about planar graphs to the subclass of graphs in $\mathcal{C}4$ that have genus g for any $g \geqslant 0$. We also discuss the shortness parameters of graphs with

[2]This settles [12, Conjecture 4]. There is a minuscule improvement of this lower bound to $\mathrm{circ}(G) \geqslant \frac{3}{4}n + 1$ in [31] and, as far as we know, no better bound has been published.

[3]In [30], Zaks claims that $\rho(\mathcal{C}4\mathcal{P}) \leqslant \frac{38}{39}$ has essentially been shown by Faulkner and Younger in [8] employing their graphs M_k; we do not see that these graphs imply the claimed bound (see [17] for more details). We will, however, show in Section 3.1 how one can use the Faulkner-Younger graph to prove $\rho(\mathcal{C}4\mathcal{P}) \leqslant \frac{39}{40}$.

large independent sets. We apply it to prove that the shortness exponent of 5-connected 1-planar graphs is strictly less than 1.

A *fragment* of a graph G is a subgraph of G along with some half-edges of G. If a fragment has k half-edges, we call it a *k-leg fragment* (see Figure 1 for an example; the dotted line splits the graph into two 4-leg fragments). For vertices x and y, we call a path between x and y an *xy-path*; this notation is extended to objects other than vertices, for instance edges and half-edges. A face of length k in a plane graph is called a *k-face*. We will make tacit use of the Jordan Curve Theorem.

2 Upper Bounds for the Shortness Coefficient of $\mathcal{C}4\mathcal{P}$

Grünbaum and Malkevitch [11] extracted a 38-vertex fragment from the 42-vertex Grinberg graph [10] by deleting the vertices of its 4-face, and then constructed a 154-vertex 4-leg fragment by adding two vertices to four copies of the 38-vertex fragment. They showed that if a graph G has a cycle C and G contains a copy of the 154-vertex fragment that does not fully contain C, then C contains at most 152 of the 154 vertices of that fragment. This implies $\rho(\mathcal{C}4\mathcal{P}) \leqslant \frac{152}{154} = \frac{76}{77}$, as then for any 4-regular 4-connected planar graph (of which there are infinitely many), we can replace every vertex with a copy of the aforementioned fragment, which gives a graph in $\mathcal{C}4\mathcal{P}$.

2.1 A 38-Vertex Fragment

We follow a similar strategy, but instead use the 38-vertex 4-leg fragment F obtained by deleting the outer 4-face of H given in Figure 1, which is considerably smaller than the 154-vertex 4-leg fragment used by Grünbaum and Malkevitch. We found F by an exhaustive computer search. For this, we used `plantri` [3] to generate cyclically 4-edge-connected cubic plane graphs, and searched for a graph H that contains a 4-face $abcd$ (cyclically counterclockwise labeled) such that $H - a$, $H - d$, $H - a - b$, $H - c - d$ and $H - ab - cd$ are non-Hamiltonian. The program determined that the smallest such graphs have 42 vertices, and that there are exactly 15 such graphs on 42 vertices. One of these graphs is shown in Figure 1. We proceed with a proof that this graph H has indeed the stated properties.

2.2 Non-Hamiltonicity Properties

The 4-leg fragment F consists of three smaller 4-leg fragments, two of which are mirror-symmetric (the two bottom ones, see Figure 1). Given the graphs H_1 and H_2 as in Figure 2, we define these smaller 4-leg fragments as follows. Let F_1 and F_2 be the 4-leg fragments obtained by deleting the outer 4-faces of H_1 and H_2, respectively. We first consider several non-Hamiltonicity properties of the graphs H_1 and H_2. We then deduce non-Hamiltonicity properties of the graph H from the non-Hamiltonicity properties of F_1 and the two copies of F_2 in H.

Lemma 1. *The graphs $H_1 - c_1 - d_1$ and $H_1 - a_1b_1 - c_1d_1$ are non-Hamiltonian.*

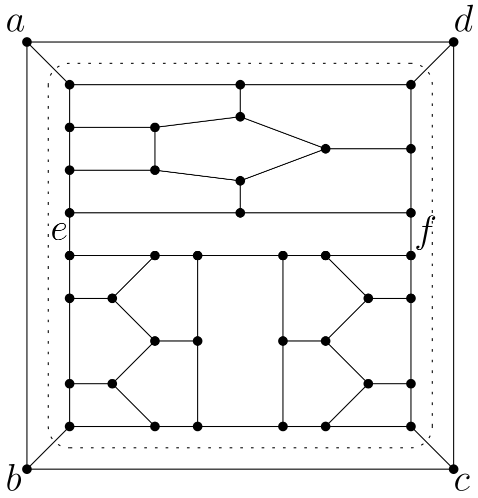
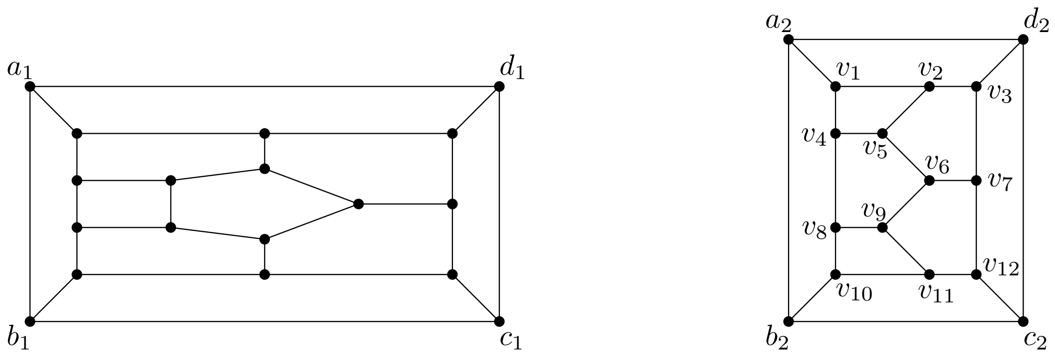


Figure 1: The 42-vertex graph H that consists of an outer 4-face and the 38-vertex 4-leg fragment F inside this 4-face.



(a) H_1

(b) H_2

Figure 2: The graphs H_1 and H_2.

Proof. We prove the lemma by Grinberg's criterion [10]. Consider the planar embedding of H_1 given in Figure 2a. Then $H_1 - c_1 - d_1$ has one 4-face, five 5-faces, one 6-face and one 9-face. Suppose to the contrary that there is a Hamiltonian cycle $\mathfrak{h}$ in $H_1 - c_1 - d_1$. Then, by Grinberg's criterion, we have

$$2(\varphi_4' - \varphi_4'') + 3(\varphi_5' - \varphi_5'') + 4(\varphi_6' - \varphi_6'') + 7(\varphi_9' - \varphi_9'') = 0,$$

where φ_k' and φ_k'' are the numbers of k-faces on the inside and on the outside of $\mathfrak{h}$ (henceforth, 'inside' and 'outside' refer to $\mathfrak{h}$ considered in the embedding). As both a_1 and b_1 have degree two, a_1b_1 is contained in $\mathfrak{h}$. Thus, the 6-face must be inside and the 9-face outside $\mathfrak{h}$, and we deduce that

$$2(\varphi_4' - \varphi_4'') \equiv 0 \pmod 3.$$

But this is clearly impossible, because the value of the left-hand side is 2 or -2.

Similarly, the graph $H_1 - a_1b_1 - c_1d_1$ has one 4-face, seven 5-faces and one 11-face. Suppose to the contrary that it contains a Hamiltonian cycle. By Grinberg's criterion, we have

$$2(\varphi_4' - \varphi_4'') + 3(\varphi_5' - \varphi_5'') + 9(\varphi_{11}' - \varphi_{11}'') = 0.$$

We deduce that

$$2(\varphi_4' - \varphi_4'') \equiv 0 \pmod 3,$$

which is impossible for the same reason as before. □

Lemma 2. *The graphs $H_2 - a_2 - b_2$, $H_2 - b_2 - c_2$ and $H_2 - a_2 - d_2$ are non-Hamiltonian.*

Proof. Consider the planar embedding of H_2 given in Figure 2b. The graph $H_2 - a_2 - b_2$ has two 4-faces, four 5-faces and one 10-face. Suppose to the contrary that it contains a Hamiltonian cycle. By Grinberg's criterion, we have

$$2(\varphi_4' - \varphi_4'') + 3(\varphi_5' - \varphi_5'') + 8(\varphi_{10}' - \varphi_{10}'') = 0.$$

We deduce that

$$2(\varphi_4' - \varphi_4'') + 8(\varphi_{10}' - \varphi_{10}'') \equiv 0 \pmod 3,$$

which is impossible, since the 10-face is outside the Hamiltonian cycle and the two 4-faces are both inside (as the vertices c_2 and d_2 have degree two in $H_2 - a_2 - b_2$).

For the graph $H_2 - b_2 - c_2$ we employ a direct argument. Suppose there is a Hamiltonian cycle $\mathfrak{h}$ in $H_2 - b_2 - c_2$. Since the vertices a_2, d_2, v_{10} and v_{12} (in the notation of Figure 2b) have degree two in this graph, $\mathfrak{h}$ must contain $v_1a_2d_2v_3$ and $v_7v_{12}v_{11}v_{10}v_8$ as subpaths. Since the edges v_9v_{11}, v_6v_7 and v_4v_8 are not contained in $\mathfrak{h}$, $\mathfrak{h}$ must contain $v_8v_9v_6$, v_3v_7 and $v_5v_4v_1$ as subpaths. Altogether we know that $\mathfrak{h}$ contains $v_5v_4v_1a_2d_2v_3v_7v_{12}v_{11}v_{10}v_8v_9v_6$ as subpath and hence $\mathfrak{h}$ does not contain v_2, which violates that $\mathfrak{h}$ is Hamiltonian. By symmetry of H_2, this gives the same claim for the graph $H_2 - a_2 - d_2$. □

We use the preceding lemmas to prove non-Hamiltonicity properties of H.

Lemma 3. *The graphs $H - a$, $H - d$, $H - a - b$, $H - c - d$ and $H - ab - cd$ are non-Hamiltonian.*

Proof. Suppose to the contrary that $H - a$ contains a Hamiltonian cycle $\mathfrak{h}$. Then $\mathfrak{h}$ contains the edges bc and cd and therefore exactly one of the edges e and f. If $\mathfrak{h}$ contains e, then some vertex of the right-hand side copy of F_2 is not contained in $\mathfrak{h}$, since $H_2 - a_2 - b_2$ is non-Hamiltonian by Lemma 2 (recall that the right-hand side copy is mirrored which switches $\{a_2, b_2\}$ and $\{c_2, d_2\}$). If $\mathfrak{h}$ contains f, then the vertices of one of the copies of F_2 are not contained in $\mathfrak{h}$, since $H_2 - b_2 - c_2$ and $H_2 - a_2 - d_2$ are non-Hamiltonian by Lemma 2. Hence, $H - a$ is not Hamiltonian. By the same argument, the graphs $H - d$, $H - a - b$ and $H - c - d$ are non-Hamiltonian.

The graph $H - ab - cd$ is non-Hamiltonian, because $H_1 - a_1b_1 - c_1d_1$ is non-Hamiltonian by Lemma 1. □

Hence, if G is a graph that contains the 4-leg fragment F and a cycle C such that $V(C) \supset V(H)$, then Lemma 3 ensures that $C \cap F$ consists of an e_ae_d-path of F or an e_be_c-path of F or both, where e_a, e_b, e_c, e_d are the half-edges in F incident to a, b, c, d in H, respectively.

2.3 A Cyclic Embedding

Let G_k be the graph obtained from linking k copies of F in a cyclic way as shown in Figure 3, which is an approach already used by Faulkner and Younger [8]. In every copy of F (see Figure 1), the edges e_a and e_b are on the outer cycle, while the edges e_c and e_d lie on the inner cycle. It is not difficult to check that G_k is in $\mathcal{C}4\mathcal{P}$, as H is in $\mathcal{C}4\mathcal{P}$. Let C be a longest cycle of G_k.

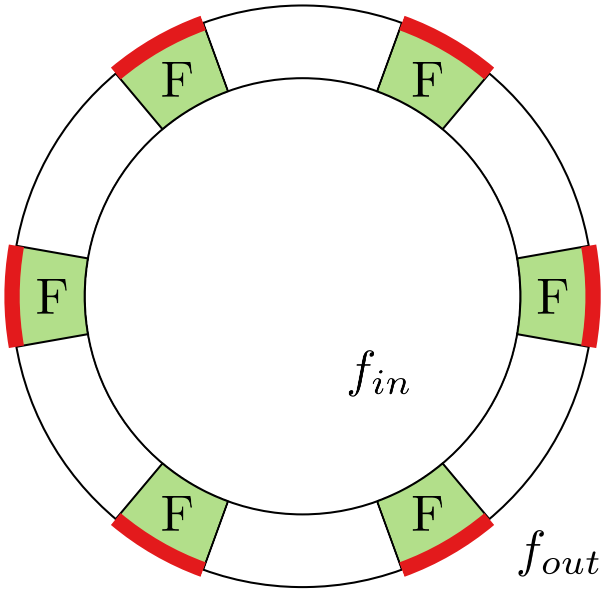


Figure 3: A cyclic embedding of copies of F. Red line segments indicate boundaries between e_a and e_b.

If the faces f_{in} and f_{out} of G_k are on the same side of C (that is, in the same region of $\mathbb{R}^2 \setminus C$), then we call C a *sausage*. If C is a sausage, every edge pair between two adjacent copies of F has the property that either both edges are in C or none of them is in C. Since C has maximal length, the latter case can happen at most once. Therefore, every copy of F up to two exceptional copies intersects with C in the union of an e_ae_b-path and an e_ce_d-path of F. By Lemma 3, this implies that C does not contain $k-2$ vertices of G_k.

If C is not a sausage, then f_{in} and f_{out} lie on different sides of C. Then C contains exactly one edge from every edge pair between two consecutive copies of F, and thus C intersects every copy of F in one e_1e_2-path of F, where $e_1 \in \{e_a, e_d\}$ and $e_2 \in \{e_b, e_c\}$. By Lemma 3, this implies that C misses at least one vertex in every copy of F (by maximality, exactly one) and therefore does not contain k vertices of G_k.

Since F has 38 vertices, the shortness coefficient of this infinite subclass of $\mathcal{C}4\mathcal{P}$ is $\frac{37}{38}$, which gives the following theorem.

Theorem 4. *The shortness coefficient of the class of cyclically 4-edge-connected cubic planar graphs is at most* $\frac{37}{38}$.

We can use the same circular arrangement to also give a bound for graphs of arbitrary genus. Denote by ρ_g the shortness coefficient of the class of cyclically 4-edge-connected cubic graphs of genus g. For 4-faces $F = v_0v_1v_2v_3$ and $F' = v_0'v_1'v_2'v_3'$ of disjoint embedded graphs, we say that we *connect* F with F' when we take the midpoints m_i of v_iv_{i+1} and the midpoints m_i' of $v_i'v_{i+1}'$, indices mod 4, and join by an edge m_i with m_i' for all i.

Theorem 5. *For all $g \geqslant 0$ we have $\rho_g \leqslant \frac{37}{38}$.*

Proof. Consider the 4-regular 4-connected toroidal graph $C_p \square C_p$ (the Cartesian product of two cycles; p at least 6). Expand each vertex into a 4-cycle. We obtain the 3-regular cyclically 4-edge-connected toroidal graph A_1 containing a 4-face Q. Let A_1' be a copy of A_1 with Q' denoting the copy of Q. Connect Q with Q' as defined above. We obtain the 3-regular cyclically 4-edge-connected genus-2 graph A_2. Iterating this procedure (clearly sufficiently many distant 4-faces are present) we construct the 3-regular cyclically 4-edge-connected genus-k graph A_k containing a 4-face R. It is clear that A_k has genus at most k, since we construct it with an embedding that has this genus. That the genus is indeed equal to k follows from the fact that one can easily find a subdivision of $K_{3,3}$ in each copy of A_1 such that all k subdivisions are pairwise vertex-disjoint, and that the genus is additive over connected components. If we attach half-edges to the midpoints of the edges of R we get a 4-leg fragment F^k with genus k.

We use a circular arrangement as in Figure 3, but this time we insert one copy of F^g, and for the rest we still use the fragment F. If there are n copies of F, we call the resulting graph $G_{g,n}$. Note that $G_{g,n}$ has genus g. So we obtain a family of graphs with genus g for which the ratio of the circumference and the order goes to $\frac{37}{38}$ if the order goes to infinity. ☐

2.4 Bounded Face Lengths

The length of the largest face in the graph class G_k that we constructed for Theorem 4 tends to infinity. Here, we show that $\mathcal{C}4\mathcal{P}$ contains a subclass of graphs whose face lengths are bounded from above by a constant, so that the shortness coefficient of this subclass is not much larger than $\frac{37}{38}$.

The following results about such graph families are known. For $t \in \{4, 5\}$, let $\mathcal{C}t\mathcal{P}(p, q)$ be the subclass of graphs in $\mathcal{C}t\mathcal{P}$ all of whose face lengths are either p or q. Zaks [29] showed that for all $k \geqslant 2$ we have $\rho(\mathcal{C}4\mathcal{P}(5, 5k + 5)) \leqslant \frac{100k+9}{100k+10}$, $\rho(\mathcal{C}4\mathcal{P}(5, 5k + 17)) < 1$ and $\rho(\mathcal{C}4\mathcal{P}(5, 13)) < 1$. Walther [28] showed the existence of an infinite family of non-Hamiltonian connected cyclically 5-edge-connected cubic planar graphs all of whose face lengths are either 5 or 8, and also proved the stronger result that $\rho(\mathcal{C}5\mathcal{P}(5, 8)) < 1$.

Theorem 6. *Let $g \geqslant 0$ and $\ell \geqslant 23$. The shortness coefficient of the class of cyclically 4-edge-connected cubic graphs of genus g and with faces of length at most ℓ is at most $\frac{45}{46}$.*

Proof. We first handle the planar case, i.e. the case where $g = 0$. Consider the 50-vertex graph H_3 of Figure 4. We remark that a computer search proved that H_3 is the smallest cyclically 4-edge-connected cubic plane graph containing a 4-face $a_3b_3c_3d_3$ (labels given in

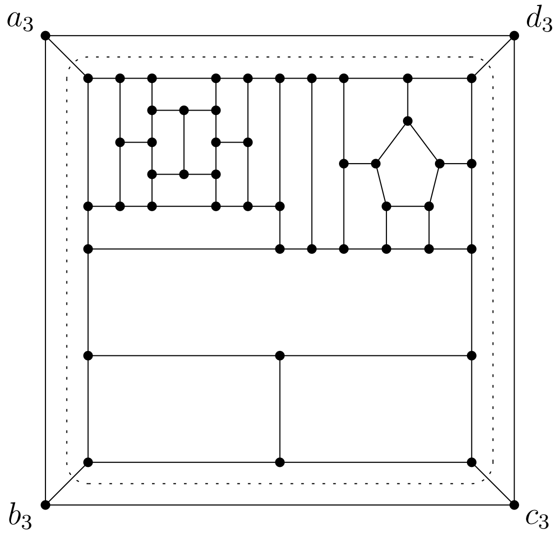


Figure 4: The 50-vertex graph H_3 and the 46-vertex 4-leg fragment F_3 that is obtained from H_3 by deleting a_3, b_3, c_3 and d_3.

cyclic order) such that H_3, $H_3 - a_3$ and $H_3 - b_3$ are non-Hamiltonian (the latter properties are proven similarly as Lemmas 1 and 2).

Let F_3 be the 4-leg fragment that is obtained from H_3 by deleting the four vertices a_3, b_3, c_3, d_3 on its outer face (each leaving a half-edge). Let e_{a_3}, e_{b_3}, e_{c_3} and e_{d_3} be the half-edges of F_3 that are incident to a_3, b_3, c_3 and d_3, respectively. The number of vertices on the clockwise boundary of F_3 between e_{a_3} and e_{d_3} is 10, between e_{d_3} and e_{c_3} is 5, between e_{c_3} and e_{b_3} is 3, and between e_{b_3} and e_{a_3} is 5.

For $k \geqslant 0$, let O_k be the graph of an octahedron with k additional bands of quartic vertices, that is, O_k consists of $4(k+1)+2$ vertices, denoted by $s, t, u_{i,j}$ for $i \in \{0, \ldots, k\}$ and $j \in \{1, 2, 3, 4\}$, and $8(k+1)+4$ edges, such that $u_{i,1}u_{i,2}u_{i,3}u_{i,4}$ is an induced 4-cycle for every $i \in \{0, \ldots, k\}$ and $su_{0,j} \ldots u_{k,j}t$ is an induced path of order $k+3$ for every $j \in \{1, 2, 3, 4\}$. Then O_0 is the octahedron graph, and, for every $k \geqslant 1$, O_k is a 4-regular 4-connected graph in which all faces are triangular or quadrangular.

Then, for any $k \geqslant 0$, the graph obtained from O_k by replacing every vertex with a copy of F_3 is such that every longest cycle misses at least one vertex of every copy of F_3, because H_3, $H_3 - a_3$, and $H_3 - b_3$ are non-Hamiltonian. One can easily verify that replacing the vertices can be done in such a way that the largest faces in the resulting graph have size at most 23, see Figure 5 for an example.

The families of graphs for genus $g \geqslant 1$ are obtained by not replacing the vertex s by a copy of F_3, but by a copy of F^g. When this is done for the configuration shown in Figure 5, this does not increase the maximum face size. □

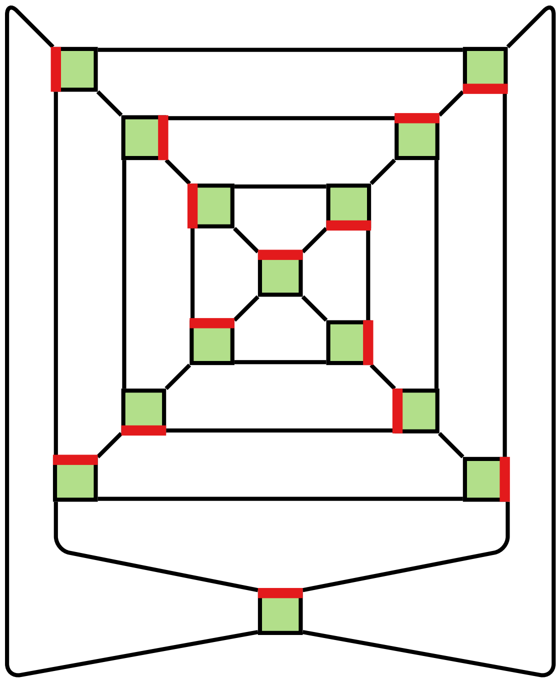


Figure 5: The graph obtained by replacing every vertex of O_2 by a copy of F_3. Red edges indicate clockwise boundaries between e_{a_3} and e_{d_3}, which contain 10 vertices.

3 Upper Bounds for the Shortness Coefficient of $\mathcal{C}4$

3.1 General Cubic Graphs

We first extend the above results by using a similar approach to obtain a general upper bound for $\rho(\mathcal{C}4)$.

Theorem 7. *Let G be a cyclically 4-edge-connected cubic n-vertex graph. Then* $\rho(\mathcal{C}4) \leqslant \frac{\text{circ}(G)-2}{n-2}$*, and if there exist adjacent vertices v, w in G such that $G - v - w$ is planar, then* $\rho(\mathcal{C}4\mathcal{P}) \leqslant \frac{\text{circ}(G)-2}{n-2}$.

Proof. Let xy be an edge of G. We see $G - x - y$ as a fragment F with legs a, b, c, d, where a and d were incident with x (in G) and b and c were incident with y. We adapt a definition of Chvátal [6] and call a pair (v, w) of legs of F *good* if there exists a vw-path in F on at least $\text{circ}(G) - 1$ vertices. A pair of pairs $((v, w), (v', w'))$ of legs of F is said to be *good* if there exist two disjoint paths P_1 and P_2 in F, one between v and w and one between v' and w', such that $|V(P_1)| + |V(P_2)| \geqslant \text{circ}(G) - 1$.

Consider the pair (a, b) and assume it to be good. Then G contains an ab-path on at least $\text{circ}(G) - 1$ vertices, which does not visit the vertices x and y. Joining the legs a and b via x and y, we obtain a cycle in G of length at least $\text{circ}(G) + 1$, an obvious contradiction. So (a, b) is not good. The pairs $(a, c), (b, d), (c, d)$ are dealt with analogously. Now consider the pair of pairs $((a, b), (c, d))$ and suppose it is good. Then G contains an ab-path P_1 and a cd-path P_2 such that $P_1 \cap P_2 = \varnothing$ and $|V(P_1)| + |V(P_2)| \geqslant$

$\operatorname{circ}(G) - 1$. Taking $P_1 \cup P_2$ as subgraph of G and joining the legs a and d via x, as well as b and c via y, we obtain a cycle in G of length at least $\operatorname{circ}(G) + 1$, once more a contradiction. The case $((a,c),(b,d))$ is analogous. We conclude that none of the pairs $(a,b),(a,c),(b,d),(c,d),((a,b),(c,d)),((a,c),(b,d))$ is good.

As depicted in Figure 3, we cyclically arrange k copies of F such that leg a (of copy ℓ) and leg b (of copy $\ell+1$), as well as leg d (of copy ℓ) and leg c (of copy $\ell+1$) are joined. We obtain a graph O that is obviously cubic, and planar if F is planar. The proof that O is cyclically 4-edge-connected is straightforward but tedious and therefore omitted.

Consider a cycle C in O and the intersection $I = C \cap F$, where F is an arbitrary copy of the above fragment residing in O. Furthermore, we assume that C is not fully contained in F, and that C visits at least $\operatorname{circ}(G) - 1$ vertices of F. If I is composed of one component P, P is either a bc-path or an ad-path, as $(a,b),(a,c),(b,d),(c,d)$ are not good. If I consists of two disjoint components P_1 and P_2, P_1 is an ad-path and P_2 is a bc-path, since $((a,b),(c,d)),((a,c),(b,d))$ are not good. Hence, a longest cycle in O misses in each of at least $k-2$ copies of F at least $n - \operatorname{circ}(G)$ vertices. □

We apply Theorem 7 to the Petersen graph and the 42-vertex Faulkner-Younger graph [8] in order to obtain:

Corollary 8. *$\rho(\mathcal{C}4) \leqslant \frac{7}{8}$ and $\rho(\mathcal{C}4\mathcal{P}) \leqslant \frac{39}{40}$.*

Note that the bound on $\rho(\mathcal{C}4\mathcal{P})$ is slightly weaker than what we gave in Theorem 4. The bound on $\rho(\mathcal{C}4)$ was due to Máčajová and Mazák [19] which improved a bound by Hägglund who—as Markström wrote in [18, p. 2]—indirectly proved in [13] that $\rho(\mathcal{C}4) \leqslant \frac{14}{15}$. In fact, applying Theorem 7 to the Petersen graph precisely gives us the graph class which was constructed by Máčajová and Mazák:

Corollary 9. *There are infinitely many cyclically 4-edge-connected cubic n-vertex graphs G with $\operatorname{circ}(G) \leqslant \frac{7}{8}n$.*

3.2 Graphs with Large Independent Sets

We end this paper with an extension of a technique used in the proof of a recent theorem of Fabrici et al. [7]. Given a graph having a large independent set, we construct a sequence of graphs and prove an upper bound of its shortness exponent.

Let G be a graph and $U \subset V(G)$ be an independent set such that each vertex v in U has degree d. Now we fix a vertex $w \in U$ and obtain a d-leg fragment F by deleting w and its incident half-edges. Vertices from $S := U - w$ are called *special*. Starting with $G_0 := G$, we construct an infinite sequence $\mathcal{G}_{G,S} = (G_k)_{k \geqslant 0}$ of graphs as follows. Let G_k be as already constructed and obtain G_{k+1} from G_k by replacing each special vertex of G_k with a copy of F. Set the special vertices of G_{k+1} to be those from each copy of F. The family $\mathcal{G}_{G,S}$ inherits various properties from G such as planarity, regularity and connectivity.

Theorem 10. *Let $d \geqslant 3$ and G be a 2-connected $(n+1)$-vertex graph containing an independent set $U \subset V(G)$ and each vertex in U has degree d. Let $w \in U$ be the vertex* *to be deleted to obtain an n-vertex d-leg fragment F, and $S := U - w$ be the set of special vertices. If $\frac{n}{2} < |S| < n$, we have*

$$\rho(\mathcal{G}_{G,S}) = 0 \text{ and } \sigma(\mathcal{G}_{G,S}) \leqslant \frac{\log(n - |S|)}{\log |S|}.$$

Proof. Let T_k be a longest closed trail of G_k visiting each non-special vertex of G_k at most once. Put $n_k := |V(G_k)|$ and $t_k := |V(T_k)|$. Since a longest cycle of G_k is also a closed trail of G_k, we have $\operatorname{circ}(G_k) \leqslant t_k$ for every $k \geqslant 0$. We denote by u the number of non-special vertices in the fragment F obtained from G. Since G_{k-1} can be obtained from G_k by contracting the copies of F into special vertices, the trail T_k will be contracted to be a trail of length at most t_k/u. This implies that $t_k \leqslant u \cdot t_{k-1}$, and hence

$$t_k \leqslant u^k \cdot t_0.$$

Furthermore,

$$n_k = 2 + (n-1) \cdot \sum_{j=0}^{k} |S|^j = 2 + (n-1) \cdot \frac{|S|^{k+1} - 1}{|S| - 1} > |S|^{k+1}.$$

Therefore,

$$\sigma(\mathcal{G}_{G,S}) \leqslant \lim_{k\to\infty} \frac{\log t_k}{\log n_k} \leqslant \lim_{k\to\infty} \frac{\log u + \frac{1}{k}\log t_0}{(1+\frac{1}{k})\log |S|} = \frac{\log u}{\log |S|}.$$

By the assumption, $0 < u < |S|$, hence we have that $\sigma(\mathcal{G}_{G,S})$ is bounded above by some constant less than 1, which implies that $\rho(\mathcal{G}_{G,S}) = 0$. $\square$

In [7] it was shown that there exists a 5-connected 1-planar graph G_0 to which we can apply Theorem 10, so the shortness coefficient of the 5-connected 1-planar graphs is 0. (For the definition of "1-planar" graphs, we refer to [7].) Furthermore, there exists no planar cubic cyclically 5-edge-connected graph satisfying the conditions stated in Theorem 10, since if there would be, we would have $\rho(\mathcal{C}5\mathcal{P}) = 0$, which is false, as by Tutte's theorem [26] we have $\rho(\mathcal{C}5\mathcal{P}) \geqslant \frac{3}{4}$.

Acknowledgements

The authors wish to thank Gunnar Brinkmann, Jan Kynčl and two anonymous referees for comments which improved the presentation of our results.

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